

Manifolds

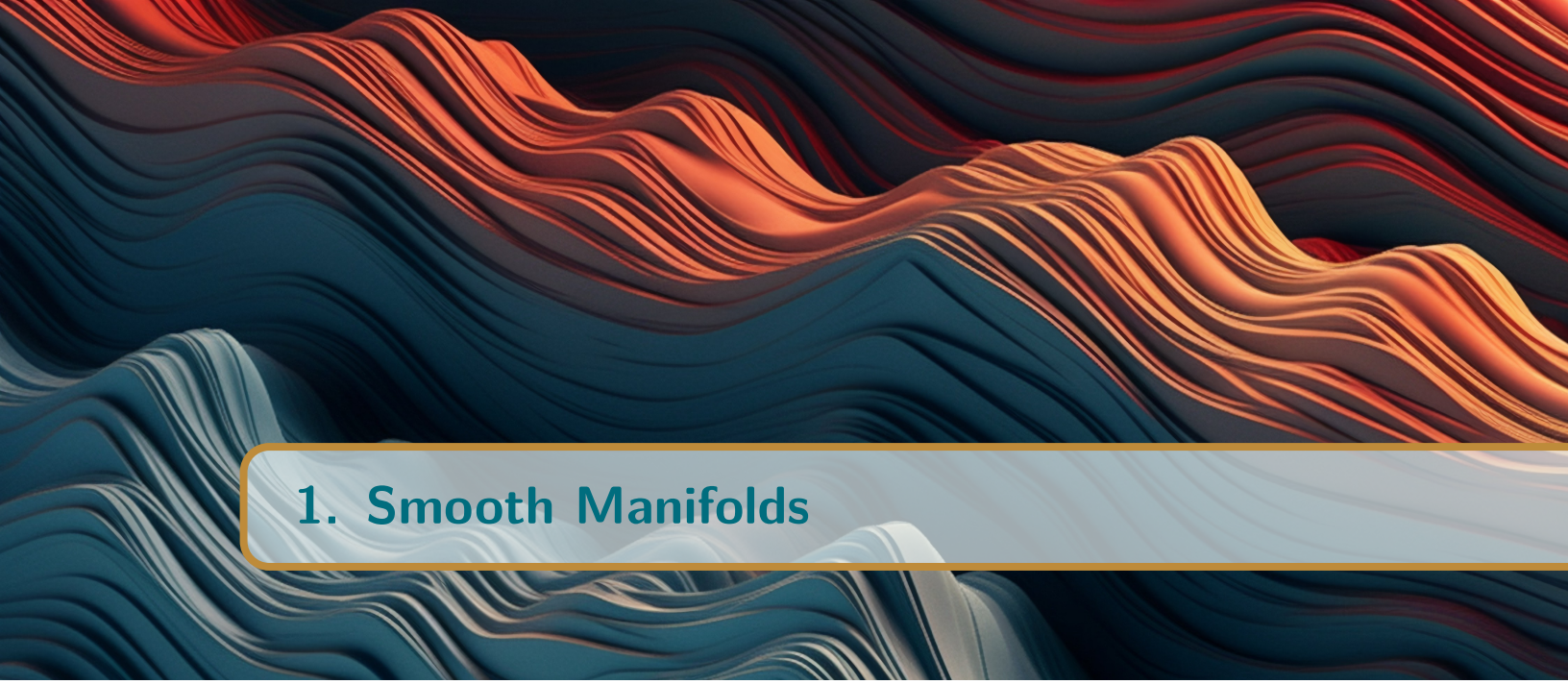
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March 7, 2026



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1. Smooth Manifolds

1.1 Problems

■ **Problem 1.1** Let X ge the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X be the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second countable, but not Hausdorff. This space is called the line with two origins.

Solution Denote the topological space $Y = X/\sim$. The open sets in Y are those that their pre-image under the quotient map $\pi : X \mapsto X/\sim$ is open. Denote the open sets around the origins as I, I_+, and I_-, where $\pi((0, -1)) \in I_-$, $\pi((0, 1)) \in I_+$, and bot holds for I.

■ **Problem 1.2 — Exercise 1.3.** A logarithmic spiral means a plane curve of the form

$$\gamma(t) = c(e^{\gamma t}\cos(t), e^{\gamma t}\sin(t)), \quad t \in \mathbb{R},$$

where $c, \lambda \in \mathbb{R}$ with $c \neq 0$. Use an improper integral to prove that a restriction of this curve to $[0, \infty)$ has finite arc length, even though it makes infinitely many loops around the origin.

Solution Consider

$$\gamma'(t) = ce^{\lambda t}(\lambda\cos(t) - e^{\lambda t}\sin(t), \lambda\sin(t) + e^{\lambda t}\cos(t))$$

Observe that

$$\|\gamma'(t)\| = ce^{\lambda t}(\lambda^2 + 1).$$

So the length of this curve will be

$$\int_0^\infty \|\gamma'(t)\|dt = \lambda c(\lambda^2 + 1) < \infty,$$

for $\lambda < 0$.

■ **Problem 1.3 — Exercise 1.4.** Consider the curve

$$\gamma(t) = (\sin(t), \cos(t) + \ln\Big(\tan\big(t/2\big)\Big)), \qquad t \in (\pi/2, \pi).$$

Demonstrate that for every point p of its image, the segment of the tangent line at p between p and the y-axis has length 1. This curve is called a tractrix.

Solution Consider

$$\begin{aligned}
\gamma'(t) &= (\cos t, -\sin t + \frac{1/2\sec^2(t/2)}{\tan\big(t/2\big)}) \\
&= (\cos t, -\sin t + \frac{\cos\big(t/2\big)}{2\cos^2(t/2)\sin\big(t/2\big)}) \\
&= (\cos t, -\sin t + \frac{1}{\sin t}) \\
&= (\cos t, \frac{\cos t}{\tan t}).
\end{aligned}$$

Consider the line $\ell(s)$ tangent to $\gamma(t)$ given by

$$\ell(s) = \gamma(t) + \gamma'(t)s.$$

Let s^* be such that $\ell(s^*) = (0,1)^T$. Length of this line segment will be

$$L = \int_0^{s^*} \big\|\ell'(s)\big\| ds = \int_0^{s^*} \big\|\gamma'(t)\big\| ds = |s^*| \cdot \big\|\gamma'(t)\big\|.$$

Observer that

$$\big\|\gamma'(t)\big\| = \Big(\frac{1}{\tan^2 t}\Big)^{1/2} = \frac{1}{\tan t}.$$

On the other hand, using $\ell(s^*) = (0,1)^T$ we can write

$$\sin t + s^* t \cos t = 0.$$

So $|s^*| = \tan t$. This implies

$$L = 1.$$