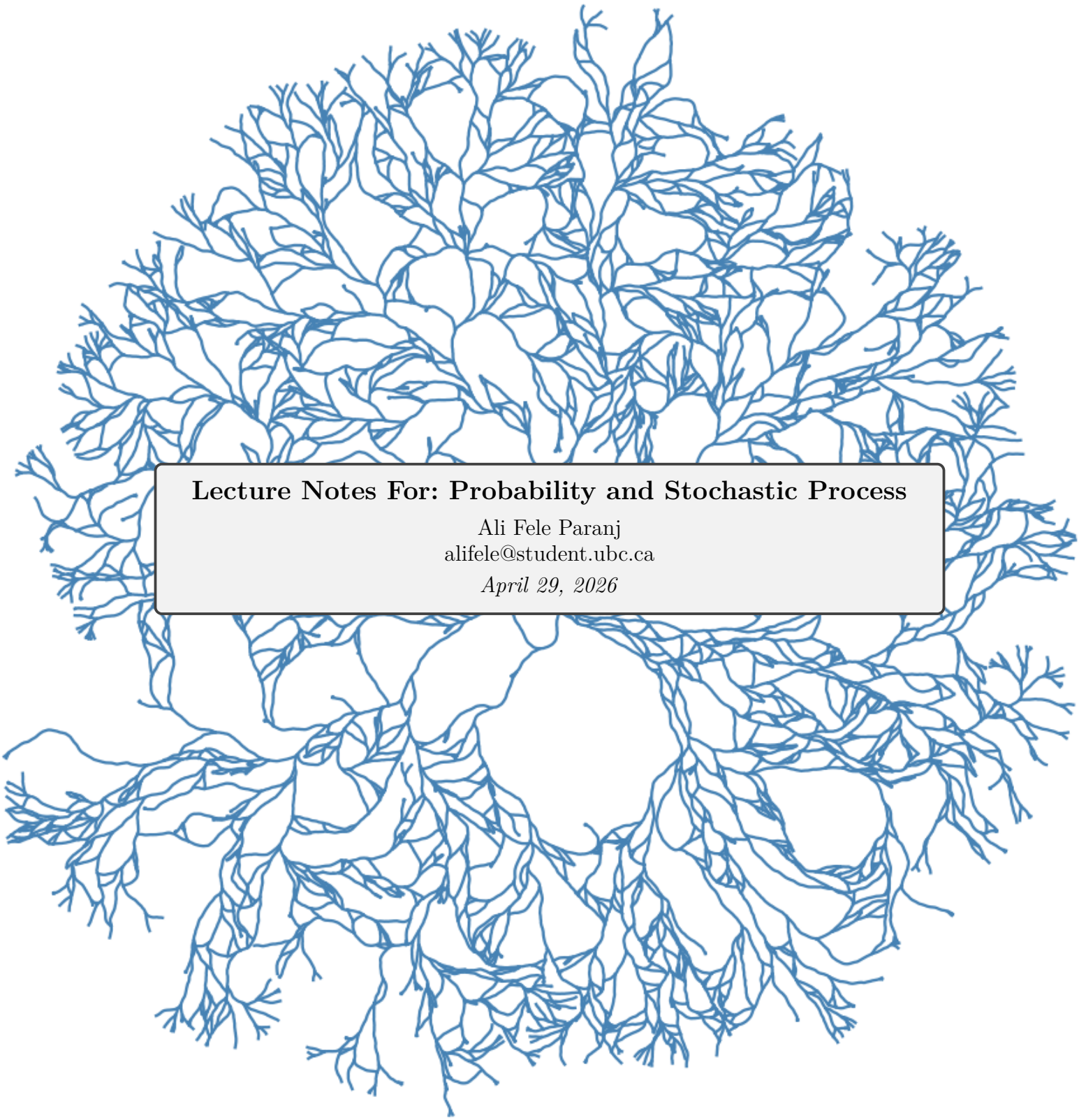


Lecture Notes For: Probability and Stochastic Process

Ali Fele Paranj
alifele@student.ubc.ca

April 29, 2026



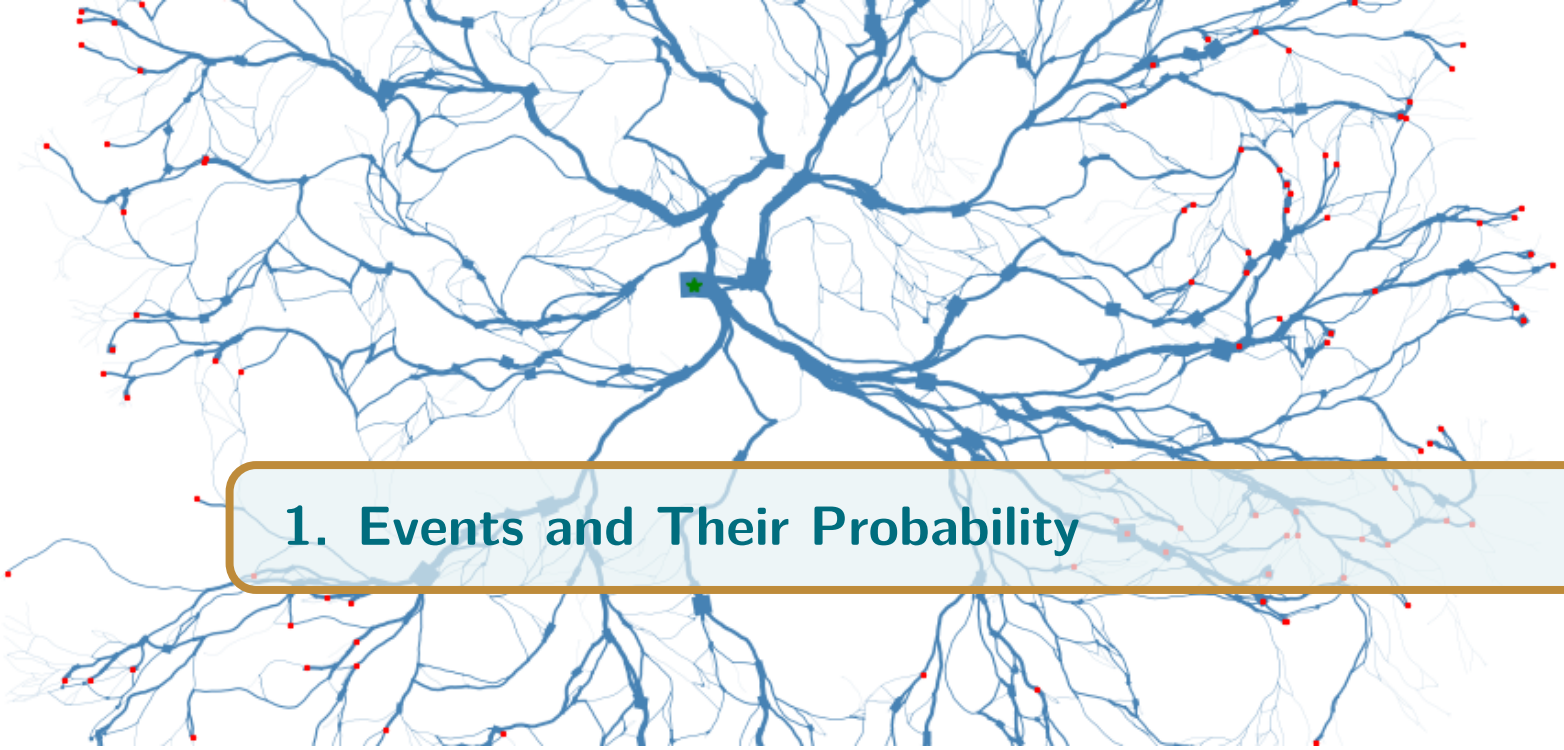
0. Contents

1 Events and Their Probability **5**

1.1 Events as Sets 5

1.2 Probability 7

1.3 Conditional Probability 10



1. Events and Their Probability

1.1 Events as Sets

■ **Problem 1.1.1** Let $\{A_i : i \in I\}$ be a collection of sets. Prove "De Morgan's Laws"'

$$\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c, \qquad \left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c.$$

Solution Let $x \in (\cup_i A_i)^c$. Then $x \notin \cup_i A_i$. So $\forall i \in I$ we have $x \notin A_i$, or equivalently $x \in A_i^c$. Using the definition of intersection $x \in \cap_i A_i^c$. For the second identity,

$$\begin{aligned} x \in \left(\bigcap_i A_i\right)^c \quad &\Leftrightarrow \\ x \notin \bigcap_i A_i \quad &\Leftrightarrow \\ \exists i \in I \text{ s.t. } x \notin A_i \quad &\Leftrightarrow \\ \exists i \in I \text{ s.t. } x \in A_i^c \quad &\Leftrightarrow \\ x \in \bigcup_i A_i^c. \end{aligned}$$

■ **Problem 1.1.2** Let A and B belong to some σ-filed $\mathcal{F}$. Show that $\mathcal{F}$ contains the sets $A \cap B$, $A \backslash B$ and $A \Delta B$.

Solution Since $A, B \in \mathcal{F}$, $A^c, B^c \in \mathcal{F}$ as well. So

$$A \cap B = (A^c \cup B^c)^c \in \mathcal{F},$$

since $A^c, B^c \in \mathcal{F}$ and $\mathcal{F}$ is closed under union and complement. Furthermore

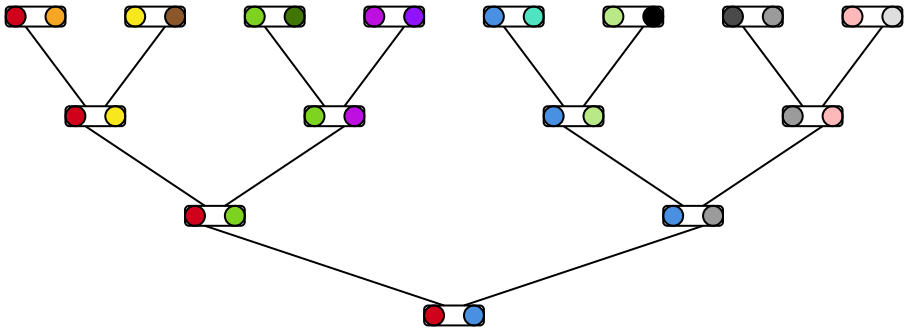
$$A \backslash B = A \cap B^c \in \mathcal{F}.$$

And

$$A \Delta B = (A \cup B) \backslash (A \cap B) \in \mathcal{F}.$$

■ **Problem 1.1.3** A conventional knock-out tournament begins with 2^n competitors and has n rounds. There are no play-offs of the positions $2, 3, \cdots, 2^n - 1$, and the initial table of the draws is specified. Gen a concise description of the sample space of all possible outcomes.

Solution Consider the following draw table for the tournament. Without loss of generality we can assume that the team on the left will win each game (as depicted in the following draw table). Then each placement of the teams in the first row represents a valid outcome of the tournament. So for 2^n players, there will be $2^n!$ possible outcomes for the tournament.



■ **Problem 1.1.4** Let $\mathcal{F}$ be a σ-field of subset of ω and suppose that $B \in \mathcal{F}$. Show that $\mathcal{G} = \{A \cap B : A \in \mathcal{F}\}$ is a σ-field of subsets of B.

Solution Any element of $\mathcal{G}$ is of the form $A \cap B$ for some $A \in \mathcal{F}$, thus $A \cap B \subset B$. It remains to show that G is a σ-field.

(a) Contains B and $\varnothing$: Because $\Omega \in \mathcal{F}$, so $\mathcal{G} \ni \Omega \cap B = B$. Furthermore, $\varnothing \in \mathcal{F}$. So $\mathcal{G} \ni \varnothing \cap B = \varnothing$.

(b) Is closed under countable union: Let $(A_i \cap B) \in \mathcal{G}$. Then

$$\bigcup_i (A_i \cap B) = (\bigcup_i A_i) \cap B \in \mathcal{G}.$$

(c) Is closed under complement with respect to B: Let $A \cap B \in \mathcal{G}$. The complement of $A \cap B$ with respect to B is

$$B \backslash (A \cap B) = B \cap (A \cap B) = B \cap (A^c \cup B) = A^c \cap B \in \mathcal{G}.$$

■ **Problem 1.1.5** Which of the following are identically true? For those that are not, say when they are true.

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(b) $A \cap (B \cap C) = (A \cap B) \cap C$.

(c) $(A \cup B) \cap C = A \cup (B \cap C)$.

(d) $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

Solution (a) True.

(b) True.

(c) Not true. Only holds when $C \subset B$.

(d) True.

1.2 Probability

■ **Problem 1.2.1** Let A and B be events with probabilities $\mathbb{P}(A) = 3/4$ and $\mathbb{P}(B) = 1/3$. Show that

$$\frac{1}{12} \leq \mathbb{P}(A \cap B) \leq \frac{1}{3}.$$

Solution Since $A \cap B \subset B$, it implies that

$$\mathbb{P}(A \cap B) \leq \mathbb{P}(B) = 1/3.$$

For the lower bound, using the inclusion exclusion principle, we have

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1 = 1/12.$$

■ **Problem 1.2.2** A fair coin is tossed repeatedly. Show that, with probability one, a head turns up sooner or later. Show similarly, that any given finite sequence of heads and tails occurs eventually with probability one. Explain the connection with Murphy's law.

Solution Let E_n be the event that at least one heads appears after n tosses.

$$\mathbb{P}(E_n) = 1 - \mathbb{P}(E_n^c) = 1 - \frac{1}{2^n}.$$

As $n \to \infty$, $\mathbb{P}(E_n) \to 1$. E_n happens with high probability.

Now we want to show that any finite sequence of heads and tails will eventually happen with probability one. Let $s = (s_1, \cdots, s_k)$ be the sequence of interest, and let E_n be the event that this sequence appears somewhere in the first $n = k \cdot N$ tosses. Observe that

$$\mathbb{P}(E_n) = 1 - \mathbb{P}(E_n^c).$$

Let B be the event that the sequence s appears somewhere in the blocks $1, \cdots, k$ or $k+1, \cdots, 2k$. Observe that $B \subset E_n$. Thus $E_n^c \subset B^c$. So

$$\mathbb{P}(E_n) = 1 - \mathbb{P}(E_n^c) \geq 1 - \mathbb{P}(B^c).$$

And for $\mathbb{P}(B)$ we have

$$\mathbb{P}(B) = (1 - \frac{1}{2^k})^N,$$

where $N = n/k$. So

$$\mathbb{P}(E_n) \geq 1 - \left(1 - \frac{1}{2^k}\right)^{n/k} = a^n, \qquad a < 1.$$

So $\mathbb{P}(E_n) \to 1$ as $n \to \infty$. So E_n happens with high probability, or equivalently, the sequence s will eventually happen.

■ **Problem 1.2.3** Six cups and saucers come in pairs: there are two cups and saucers which are red, two white, and two with starts on on. If the cups are placed randomly onto the saucer, find the probability that no cup is upon a saucer of the same pattern.

Solution Consider the events:

R : The red cup matches with the red saucer,

W : The white cup matches with the white saucer,

S : The star cup matches with the star saucer.

Then we are interested in $\mathbb{P}(S^c \cap R^c \cap W^c)$. We can write

$$\begin{aligned}\mathbb{P}(S^c \cap R^c \cap W^c) &= 1 - \mathbb{P}(S \cup R \cup W) \\ &= 1 - (\mathbb{P}(S) + \mathbb{P}(R) + \mathbb{P}(W) - \mathbb{P}(S \cap R) - \mathbb{P}(S \cap W) - \mathbb{P}(R \cap W) + \mathbb{P}(R \cap S \cap W)).\end{aligned}$$

Observer that

$$\mathbb{P}(S) = \mathbb{P}(R) = \mathbb{P}(W) = 1/3,$$

and

$$\mathbb{P}(S \cap R) = \mathbb{P}(S|R) \cdot \mathbb{P}(R) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6},$$

and with a similar calculation:

$$\mathbb{P}(S \cap R) = \mathbb{P}(S \cap W) = \mathbb{P}(R \cap W) = \frac{1}{6}.$$

And finally $\mathbb{P}(S \cap R \cap W) = \mathbb{P}(S|R \cap W) \cdot \mathbb{P}(R \cap W) = 1 \cdot \frac{1}{6}$. So

$$\mathbb{P}(S^c \cap W^c \cap R^c) = 1 - (1 - \frac{1}{2} + \frac{1}{6}) = \frac{1}{3}.$$

■ **Problem 1.2.4** Let $A_1, A_2, \cdots, A_n$ be events where $n \geq 2$, and prove that

$$\mathbb{P}(\cup_i A_i) = \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n+1} \mathbb{P}(A_1 \cap \cdots \cap A_n).$$

Also, assume that in each packet of Corn Flakes may be found a plastic bust of one of the last five Vice-Chancellors of Cambridge University, the probability that any give packet contains any specific Vice-Chancellor being 1/5, independently of all other packets. Show that the probability that each of the last three Vice-Chancellor is obtained in a bulk purchase of six packets is $1 - 3(4/5)^6 + 3(3/5)^6 - (2/5)^5$.

Solution To prove the inclusion-exclusion principle, there we can either prove it directly, or prove it for the simple case $n = 2$ and use induction. To prove directly, let $\mathbb{1}_{A_i} : \Omega \to \{0, 1\}$, where $\mathbb{1}_{A_i}(x) = 1$ if $x \in A_i$ and zero otherwise. Observe:

$$\begin{aligned}\mathbb{1}_{\cup_i A_i} &= 1 - \mathbb{1}_{\cap_i A_i^c} = 1 - \prod_i \mathbb{1}_{A_i^c} = 1 - \prod_i (1 - \mathbb{1}_{A_i}) \\ &= \sum_i \mathbb{1}_{A_i} - \sum_{i<j} \mathbb{1}_{A_i \cap A_j} + \cdots + (-1)^{n+1} \mathbb{1}_{\cap_i A_i}.\end{aligned}$$

Taking expectation we will get

$$\mathbb{P}(\cup_i A_i) = \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n+1} \mathbb{P}(A_1 \cap \cdots \cap A_n)$$

For the second part of the question, let A_i be the event that the i^{th} to last vice president does not happen. So we are interested in

$$\mathbb{P}(A_1^c \cap A_2^c \cap A_3^c).$$

Using the inclusion-exclusion principle we can write

$$\mathbb{P}(A_1^c \cap A_2^c \cap A_3^c) = 1 - \mathbb{P}(A_1 \cup A_2 \cup A_3) = 1 - (\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - \mathbb{P}(A_2 \cap A_3) + \mathbb{P}(A_1 \cap A_2 \cap A_3)).$$

Observe that

$$\mathbb{P}(A_1) = \mathbb{P}(A_2) = \mathbb{P}(A_3) = (\frac{4}{5})^6,$$

and

$$\mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1|A_2)\mathbb{P}(A_2) = (\frac{3}{4})^6 \cdot (\frac{4}{5})^6 = (\frac{3}{5})^6.$$

And finally

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1|A_2 \cap A_3)\mathbb{P}(A_2 \cap A_3) = (\frac{2}{3})^6 \cdot (\frac{3}{5})^6 = (\frac{2}{5})^6.$$

So

$$\mathbb{P}(A_1^c \cap A_2^c \cap A_3^c) = 1 - 3(4/5)^6 + 3(3/5)^6 - (2/5)^5$$

■ **Problem 1.2.5** Let A_r with $r \geq 1$, be the events such that $\mathbb{P}(A_r) = 1$ for all r. Show that $\mathbb{P}(\cap_r A_r) = 1$.

Solution First observe that for any A_i and A_j we have $\mathbb{P}(A_i \cap A_j) = 1$. This follows from:

$$1 = \mathbb{P}(A_i \cup A_j) = \mathbb{P}(A_1) + \mathbb{P}(A_j) - \mathbb{P}(A_i \cap A_j),$$

where we have used the fact that since $A_i \cup A_j \supseteq A_i$, hence $\mathbb{P}(A_i \cup A_j) \geq \mathbb{P}(A_i) = 1$, so $\mathbb{P}(A_i \cup A_j) = 1$. By using the inclusion-exclusion principle for three sets A_i, A_j, A_k, we get that

$$\mathbb{P}(A_i \cap A_j \cap A_k) = 1.$$

Let

$$B_1 = A_1, \quad B_2 = A_1 \cap A_2, \quad B_3 = A_1 \cap A_2 \cap A_3, \qquad .$$

Observe that $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$, and let $B = \cap_n B_n = \cap_n A_n$. By continuity of probability

$$\mathbb{P}(\cap_n A_n) = \mathbb{P}(B) = \lim_n \mathbb{P}(B_n) = 1,$$

where we have used the fact that $\mathbb{P}(B_n) = 1$ as shown above.

■ **Problem 1.2.6** You are given that at least one of the events A_r, $1 \leq r \leq n$, is certain to occur, but certainly no more thank two occur. If $\mathbb{P}(A_r) = p$, and $\mathbb{P}(A_r \cap A_s) = q$ with $r \neq s$, show that $p \geq 1/n$ and $q \leq 2/n$.

Solution We know that $\mathbb{P}(\cup_i A_i) = 1$ and $\mathbb{P}(A_i \cap A_j \cap A_k) = 0$. By inclusion exclusion principle we have

$$1 = \mathbb{P}(\cup_i A_i) = \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) = np - \frac{n(n-1)}{2} q.$$

Solving for p we get

$$p = \frac{1}{n} + \frac{n-1}{2} q.$$

It immediately follows that $p \geq 1/n$. Furthermore, solving for q we get

$$q = \frac{2(np-1)}{n(n-1)}.$$

Since $p \leq 1$, then it follows that $q \leq 2/n$.

■ **Remark** The idea above can be interpreted as an "almost" partitioning of Ω into equal sized sets, where two sets can have "slight" overlap (hence the word almost partition), but no three sets have overlap.

■ **Problem 1.2.7** You are given that at least one, but no more than three of the events A_r, $1 \le r \le n$, occur, where $n \ge 3$. The probability of at least two occurring is $1/2$. If $\mathbb{P}(A_r) = p$, $\mathbb{P}(A_r \cap A_s) = q$, $r \ne s$, and $\mathbb{P}(A_r \cap A_s \cap A_t) = x$, $r < s < t$, show that $p \ge 3/(2n)$, and $q < 4/n$.

Solution Still thinking!

1.3 Conditional Probability

■ **Problem 1.3.1** Prove that $\mathbb{P}(A|B) = \mathbb{P}(B|A)\mathbb{P}(A)/\mathbb{P}(B)$ whenever $\mathbb{P}(A)\mathbb{P}(B) \ne 0$. Show that if $\mathbb{P}(A|B) > \mathbb{P}(A)$, then $\mathbb{P}(B|A) > \mathbb{P}(B)$.

Solution Observe that

$$\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A)/\mathbb{P}(B).$$

Furthermore, since

$$\mathbb{P}(A|B) > \mathbb{P}(B),$$

we have

$$\mathbb{P}(A \cap B) > \mathbb{P}(A)\mathbb{P}(B),$$

and then we can write

$$\mathbb{P}(B \cap A)/\mathbb{P}(B) > \mathbb{P}(A).$$

This implies that

$$\mathbb{P}(B|A) > \mathbb{P}(A).$$

■ **Problem 1.3.2** For events $A_1, A_2, \cdots, A_n$ satisfying $\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_{n-1}) > 0$, prove that

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2)\cdots\mathbb{P}(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

Solution Observe that

$$\mathbb{P}(\cap_i^n A_i) = \mathbb{P}(A_n|\cap_{i=1}^{n-1} A_i)\mathbb{P}(\cap_{i=1}^{n-1} A_i).$$

Continuing with this expansion on the right hand side, we will get the desired identity.

■ **Problem 1.3.3** A man possesses five coins, two of which are double-headed, one is double tailed, and two are normal. He shuts his eyes, picks a coin at random, and tosses it. What is the probability that the lower face of the coin is a head? He opens his eyes and sees that the coin is showing heads. What is the probability that the lower face is a head? He shuts his eyes again and tosses the coin again. What is the probability that the lower face is a head?

He discards this coin, picks another at random, and tosses it. What is the probability that it shows heads?

Solution Let HH denote the event that the tossed coin is the head-head coin, TT the event that the tossed coin is the tail-tail coin, and N the even that the tossed coins is a normal coin. Let LH denote the event that the lower face is a head.

$$\begin{aligned}\mathbb{P}(LH) &= \mathbb{P}(LH|HH)\mathbb{P}(HH) + \mathbb{P}(LH|TT)\mathbb{P}(TT) + \mathbb{P}(LH|N)\mathbb{P}(N) \\ &= \frac{2}{5} + 0 + \frac{1}{2}\cdot\frac{2}{5} = \boxed{\frac{3}{5}}.\end{aligned}$$

For the probability of the lower face being head after observing the upper face is head, let UH denote the probability that the upper face is head. We can write

$$\mathbb{P}(LH|UH) = \mathbb{P}(LH|HH \cap UH)\mathbb{P}(HH|UH) + \mathbb{P}(LH|TT \cap UH)\mathbb{P}(TT|UH) + \mathbb{P}(LH|N \cap UH)\mathbb{P}(N|UH)$$

Observe that

$$\mathbb{P}(UH) = \mathbb{P}(UH|HH)\mathbb{P}(HH) + \mathbb{P}(UH|TT)\mathbb{P}(TT) + \mathbb{P}(UH|N)\mathbb{P}(N) = \frac{2}{5} + 0 + \frac{1}{2} \cdot \frac{2}{5} = \frac{3}{5}.$$

So we can write

$$\mathbb{P}(HH|UH) = \frac{\mathbb{P}(UH|HH)\mathbb{P}(HH)}{\mathbb{P}(UH)} = \frac{(1) \cdot (2/5)}{3/5} = \frac{2}{3}.$$

Similarly

$$\mathbb{P}(TT|UH) = \frac{\mathbb{P}(UH|TT)\mathbb{P}(TT)}{\mathbb{P}(UH)} = 0,$$

and

$$\mathbb{P}(N|UH) = \frac{\mathbb{P}(UH|N)\mathbb{P}(N)}{\mathbb{P}(UH)} = \frac{(1/2) \cdot (2/5)}{3/5} = \frac{1}{3}.$$

So

$$\mathbb{P}(LH|UH) = \frac{2}{3} + 0 + 0 = \boxed{\frac{2}{3}}.$$

For the scenario, where he tosses the coin again, it is the same as the person randomly looks at one side of the selected coin and observes that it is heads. Let this event be denoted by SH (seen heads). So

$$\begin{aligned}\mathbb{P}(LH|SH) &= \mathbb{P}(LH|SH, HH)\mathbb{P}(HH|SH) + \underbrace{\mathbb{P}(LH|SH, TT)\mathbb{P}(TT|SH)}_{0} + \mathbb{P}(LH|SH, N)\mathbb{P}(N|SH) \\ &= \frac{2}{3} + \frac{1}{2}\mathbb{P}(N|SH).\end{aligned}$$

For the second term we have

$$\mathbb{P}(N|SH) = \frac{\mathbb{P}(SH|N)\mathbb{P}(N)}{\mathbb{P}(SH)} = \frac{\frac{1}{2} \cdot \frac{2}{5}}{\frac{3}{5}} = \frac{1}{3}.$$

So

$$\mathbb{P}(LH|SH) = \frac{2}{3} + \frac{1}{6} = \boxed{\frac{5}{6}}.$$

After the play looks at the coin and finds it is a head facing up, then the probability that the lower face is head is $\mathbb{P}(LH|SH, UH)$. Note that we can write this as $\mathbb{P}_{SH}(LH|UH)$ for simpler notation.

$$\begin{aligned}\mathbb{P}_{SH}(LH|UH) &= \mathbb{P}_{SH}(LH|UH, HH)\mathbb{P}_{SH}(HH|UH) \\ &\quad + \mathbb{P}_{SH}(LH|UH, TT)\mathbb{P}_{SH}(TT|UH) + \mathbb{P}_{SH}(LH|UH, N)\mathbb{P}_{SH}(N|UH) \\ &= \mathbb{P}(HH|UH, SH),\end{aligned}$$

where we have used the fact that $\mathbb{P}_{SH}(LH|UH, TT) = \mathbb{P}_{SH}(LH|UH, N) = 0$, and $\mathbb{P}_{SH}(LH|UH, HH) = 1$. To calculate $\mathbb{P}(HH|UH, SH)$ one can write

$$\mathbb{P}_{SH}(HH|UH) = \frac{\mathbb{P}_{SH}(UH|HH)\mathbb{P}_{SH}(HH)}{\mathbb{P}_{SH}(UH)} = \frac{1 \cdot (2/3)}{5/6} = \frac{4}{5},$$

where we have used the fact that $\mathbb{P}(HH|SH) = \mathbb{P}(HH|UH)$ and $\mathbb{P}(UH|SH) = \mathbb{P}(UH|SH) = \mathbb{P}(LH|SH)$.

For the last part of the question, let the events $HH^{(1)}, TT^{(1)}, N^{(1)}$ denote the probability that the first coin was HH, TT, and N respectively. And let H denote the event that the the result of the toss is heads. Let E be the event of observations so far (after first toss the heads was up and after the second toss the heads was up.)

$$\mathbb{P}_E(H) = \mathbb{P}_E(H|HH^{(1)})\mathbb{P}_E(HH^{(1)}) + \underbrace{\mathbb{P}_E(H|TT^{(1)})}_{0}\mathbb{P}_E(TT^{(1)}) + \mathbb{P}_E(H|N^{(1)})\mathbb{P}_E(N^1).$$

TODO: solution of the last part incomplete.

■ **Remark — Important.** Note that while solving the problem above, although we were using $\mathbb{P}$ for the probability measure in all of the cases, but in different parts of the question the probability space has been implicitly changing, this $\mathbb{P}$ measure.

■ **Problem 1.3.4** What do you think of the following "proof" by Lewis Carroll that an urn cannot contain two balls of the same colour?

Suppose that the urn contains two balls, each of which is either black or white; thus, in the obvious notation,

$$\mathbb{P}(BB) = \mathbb{P}(BW) = \mathbb{P}(WB) = \mathbb{P}(WW) = \frac{1}{4}.$$

We add a black ball, so that

$$\mathbb{P}(BBB) = \mathbb{P}(BBW) = \mathbb{P}(BWB) = \mathbb{P}(BWW) = \frac{1}{4}.$$

Next we pick a ball at random; the chance that the ball is black is (using conditional probabilities)

$$1 \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} = \frac{2}{3}.$$

However, if there is probability $\frac{2}{3}$ that a ball, chosen randomly from three, is black, then there must be two black and one white, which is to say that originally there was one black and one white ball in the urn.

Solution Consider the following description of the problem: We choose two consecutive Black or White balls randomly each with probability 1/2 and put it in an urn. We add one more black ball. Then the probability of 2/3 calculated above, is the probability of observing a black, if we draw a random ball from the urn. The 2/3 probability, does not imply that the content was 2 black balls and 1 white ball. Because this number refers to a multi-stage experiment, rather than simply drawing a ball from an urn containing three balls.

■ **Problem 1.3.5 — The Monty Hall Problem: goats and cars.** (a) In a game show, you have to choose on of three doors, one conceals a new car, two conceal old goats. You choose, but your chosen door is not opened immediately. Instead the presenter opens another door, which reveals a goat. Her offers you the opportunity to change your choice to the third door that is unoppened. Let p be the conditional probability that the third door conceals the car. The presenters's protocol is:

(i) He is determined to show you a goat: with a choice of two, he picks one at random. Show $p = 2/3$.

(ii) He is determined to show you a goat: with a choice of two goats (Bill and Nan say), he shows you Bill with probability b. Shwo taht given you see bill, the probability is $1/(1+b)$.

(iii) He opens a door chosen at random irrespective if what lies behind. Show $p = 1/2$.

(b) Show that, for $\alpha \in [1/2, 2/3]$, there exists a protocol such that $p = \alpha$. Are you well advised to change your choice to the third door?

(c) In a variant of this question, the presenter is permitted to open the first door chosen, and to reward you with whatever lies behind. If he chooses to open another door, then this door invariably conceals a goat. Let p be the probability that the unopened door conceals the car, conditional on the presenter having chosen to open a second door. Devise protocols to yield the values $p = 0$, $p = 1$, and deduce that, for any $\alpha \in [0, 1]$, there exists a protocol with $p = \alpha$.